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The Moser–Veselov equation

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Abstract

We study the orthogonal solutions of the matrix equation $XJ - JX^T = M$, where J is symmetric positive definite and M is skew-symmetric. This equation arises in the discrete version of the dynamics of a rigid body, investigated by Moser and Veselov (Commun. Math. Phys. 139 (1991) 217). We show connections between orthogonal solutions of this equation and solutions of a certain algebraic Riccati equation. This will bring out the symplectic geometry of the Moser–Veselov equation and also reduces most computational issues about solutions to finding invariant subspaces of a certain Hamiltonian matrix. Necessary and sufficient conditions for the existence of orthogonal solutions (and methods to compute them) are presented. Our method is contrasted with the Moser–Veselov approach (Commun. Math. Phys. 139 (1991) 217). We also exhibit explicit solutions of a particular case of the Moser–Veselov equation, which appears associated with the continuous version of the dynamics of a rigid body.

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1. Introduction

In [13] Moser and Veselov investigated a discrete version of the Euler–Arnold equation for the motion of the generalized rigid body. The main feature of this discrete model is the connection between a skew-symmetric matrix M , representing the angular momentum, and the orthogonal matrix X representing the angular velocity.

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In this paper we present a complete study of the matrix equation

$$XJ - JX^T = M, \quad (1)$$

where J is symmetric positive definite and M is skew-symmetric.

After relating the orthogonal solutions of (1) with the symmetric solutions of a certain algebraic Riccati equation, most results follow from this well known theory, applied to our particular situation. Necessary and sufficient conditions for the existence and uniqueness of orthogonal solutions of Eq. (1) are presented, followed by computational considerations. Our method is also compared with the approach of Moser and Veselov in [13]. In the last section we derive explicit formulae for solutions of Eq. (1) when $J > 0$ is a scalar matrix. This particular case is associated with the continuous version of the dynamics of a rigid body (see [2]).

2. The algebraic Riccati equation revisited

The greatest stimulus for investigation of matrix Riccati equations in differential, difference, and algebraic forms, has been the linear quadratic regulator problem in optimal control. In the next section it will become clear the strong connection between orthogonal solutions of the Moser–Veselov equation (1) and symmetric solutions of a particular algebraic Riccati equation, so that a complete study of (1) follows from this well known theory.

We recall in this section some results which will be crucial throughout the paper. There is a vast literature on the subject, but we refer to Lancaster and Rodman [10,11] and Kucera [9], for more details about the theory of Riccati equations.

Consider an algebraic matrix Riccati equation of the form

$$SDS + SA + A^T S - C = 0, \quad (2)$$

where D , A and C are given real $(n \times n)$ matrices with $D \geq 0$ (non-negative definite) and $C = C^T$. As usual, associate to Eq. (2) the following Hamiltonian matrix:

$$\mathcal{H} = \begin{bmatrix} A & D \\ C & -A^T \end{bmatrix}. \quad (3)$$

We are particularly interested in real symmetric solutions of the matrix Riccati equation. The following theorems give necessary and sufficient conditions for the existence and uniqueness of symmetric solutions of (2).

Theorem 2.1 [10]. *If (A, D) is controllable, then the following statements are equivalent:*

- (i) *There exists a solution S of (2) such that $S = S^T$;*
- (ii) *There exists a solution S_1 of (2) such that $S_1 = S_1^T$ and $\operatorname{Re} \lambda \leq 0$ for every $\lambda \in \sigma(DS_1 + A)$ where $\sigma(X)$ denotes the spectrum of X ;*

- (iii) There exists a solution S_2 of (2) such that $S_2 = S_2^T$ and $\operatorname{Re} \lambda \geq 0$ for every $\lambda \in \sigma(DS_2 + A)$;
- (iv) The size of the Jordan blocks of \mathcal{H} , associated to the pure imaginary eigenvalues, is even. Moreover, if any of the statements (i)–(iv) holds, then
- (v) The solution S_1 of (2) with the properties from (ii) is unique;
- (vi) The solution S_2 of (2) with the properties from (iii) is unique.

Theorem 2.2 [10]. *If (A, D) is controllable then (2) has a unique symmetric solution if and only if all the eigenvalues of \mathcal{H} are pure imaginary and the associated Jordan blocks have even size.*

Now let \mathcal{C} be any subset of eigenvalues of \mathcal{H} with nonzero real part, which satisfies the following properties:

- (P1) $\lambda \in \mathcal{C} \Rightarrow \bar{\lambda} \in \mathcal{C} \text{ and } -\bar{\lambda} \notin \mathcal{C}$,
 - (P2) \mathcal{C} is maximal with respect to property (P1).
- (4)

Theorem 2.3 [11]. *If (2) has symmetric solutions, then for every set \mathcal{C} given as above, there exists a unique symmetric solution S such that \mathcal{C} is exactly the set of eigenvalues of $A + DS$ having nonzero real parts.*

Recall that a pair (A, B) of $n \times n$ matrices is said to be *stabilizable* if there exists a matrix K such that $A + BK$ is stable, that is, $\sigma(A + BK)$ lies on the open left half plane. (A, B) is said to be *detectable* if (B^T, A^T) is stabilizable.

An important necessary and sufficient condition for existence and uniqueness of a non-negative definite solution of (2) is given in the following theorem.

Theorem 2.4 [9]. *Assume that $C \geq 0$, $D \geq 0$, $C = EE^T$ with $\operatorname{rank}(E) = \operatorname{rank}(C)$ and $D = FF^T$ with $\operatorname{rank}(F) = \operatorname{rank}(D)$. Then (2) has a unique solution $S \geq 0$ such that the matrix $-(A + DS)$ is stable if and only if (A, F) is stabilizable and (E, A) is detectable.*

Theorem 2.5 [9]. *Assume that $C \geq 0$, $D \geq 0$, (A, F) is stabilizable and (E, A) is detectable. Then \mathcal{H} has no pure imaginary (or zero) eigenvalues and Eq. (2) has a unique symmetric solution $S > 0$ such that the matrix $-(A + DS)$ is stable.*

It is also widely known that, under the assumptions of the previous theorem, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of \mathcal{H} in the open left half plane, the unique positive definite solution S_* of (2) is given by $S_* = ZY^{-1}$, where Y, Z are $n \times n$ real matrices and the columns of $\begin{bmatrix} Y \\ Z \end{bmatrix}$ span the invariant subspace of \mathcal{H} associated with $\lambda_1, \dots, \lambda_n$.

3. Existence and uniqueness theorems

We start with a simple observation about general solutions of (1).

Lemma 3.1. *Every solution of (1) (not necessarily orthogonal) can be written in the form*

$$X = (M/2 + S)J^{-1},$$

where S is symmetric.

Proof. If X is a solution of (1), then there exists a symmetric matrix S such that

$$XJ + (XJ)^T = 2S.$$

Adding this equation with (1) it follows immediately that $X = (M/2 + S)J^{-1}$, with S symmetric. \square

The connection between orthogonal solutions of (1) and symmetric solutions of an algebraic matrix Riccati equation is given in the following result.

Lemma 3.2. *Every orthogonal solution of (1) can be written in the form $X = (M/2 + S)J^{-1}$, where S is a symmetric matrix satisfying*

$$S^2 + S(M/2) + (M/2)^T S - (M^2/4 + J^2) = 0. \quad (5)$$

Proof. By the previous lemma, $X = (M/2 + S)J^{-1}$, with S symmetric. Using the requirement that X is orthogonal, we may write

$$\begin{aligned} & ((M/2 + S)J^{-1})^T (M/2 + S)J^{-1} = I, \\ \Leftrightarrow & J^{-1}(S - M/2)(S + M/2)J^{-1} = I, \\ \Leftrightarrow & S^2 + S(M/2) - (M/2)S - M^2/4 = J^2, \\ \Leftrightarrow & S^2 + S(M/2) + (M/2)^T S - (M^2/4 + J^2) = 0. \quad \square \end{aligned}$$

The following theorem, which is an immediate consequence of the previous lemmas, allows the analysis of (1) using the theory of algebraic Riccati equations.

Theorem 3.3. *$X = (M/2 + S)J^{-1}$ is an orthogonal solution of (1) if and only if S is a symmetric solution of the algebraic Riccati equation (5).*

Remark 3.4. Introducing $W = XJ$, equation (1) reduces to $W - W^T = M$, with the additional constraint $W^T W = J^2$. Note that when X is orthogonal, $W = XJ$ is the polar decomposition of W . Thus, solving (1) is equivalent to finding the orthogonal factor in the polar decomposition of a matrix, knowing its skew-symmetric part only. Clearly, in general, there are infinitely many solutions.

Remark 3.5. Eq. (5) may be written in the form

$$(S + M/2)^T(S + M/2) = J^2,$$

which, in particular, implies that $0 \notin \sigma(S + M/2)$. This agrees with the form of the orthogonal solutions in Lemma 3.1.

Remark 3.6. Eq. (5) is a matrix Riccati equation of the form (2), with $D = I$ (non-negative definite), $C = M^2/4 + J^2$ (symmetric), and $A = M/2$. In this case the pair (A, D) is always controllable and the associated Hamiltonian matrix is now

$$\overline{\mathcal{H}} = \begin{bmatrix} M/2 & I \\ M^2/4 + J^2 & M/2 \end{bmatrix}.$$

Remark 3.7. A simple calculation shows that if

$$P = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix}$$

and S is a solution of (1), then

$$P^{-1}\overline{\mathcal{H}}P = \begin{bmatrix} M/2 + S & I \\ 0 & -(S + M/2)^T \end{bmatrix}, \quad (6)$$

which implies that

$$\sigma(\overline{\mathcal{H}}) = \sigma(S + M/2) \cup \sigma(-(S + M/2)^T) \quad (7)$$

and consequently,

$$\sigma(\overline{\mathcal{H}}) \cap i\mathbb{R} = \emptyset \iff \sigma(S + M/2) \cap i\mathbb{R} = \emptyset,$$

$$\sigma(\overline{\mathcal{H}}) \subset i\mathbb{R} \iff \sigma(S + M/2) \subset i\mathbb{R}.$$

Theorem 3.8 (Existence). *There exists a solution $X \in SO(n)$ (the special orthogonal group) for (1) if and only if the size of the Jordan blocks associated to the pure imaginary eigenvalues of \mathcal{H} (if any) is even.*

Proof. (\Rightarrow) Since the pair $(M/2, I)$ is controllable, the result follows by direct application of the previous theorem and the equivalence between the statements (i) and (iv) in Theorem 2.1.

(\Leftarrow) If the size of the Jordan blocks associated to the pure imaginary eigenvalues of \mathcal{H} is even, then, according to Theorem 2.1, there exists a symmetric solution S of (5) such that $\operatorname{Re} \lambda \geq 0$, for every $\lambda \in \sigma(M/2 + S)$. But this implies that $\det(S + M/2) > 0$ and, consequently, the orthogonal solution of (1), $X = (M/2 + S)J^{-1}$, has determinant equal to 1. \square

An immediate consequence of the results in this section and Theorem 2.2 is the following.

Corollary 3.9 (Uniqueness). *The matrix equation (1) has a unique solution $X \in SO(n)$ if and only if the spectrum of $\overline{\mathcal{H}}$ is pure imaginary and the size of the Jordan blocks associated to each (nonzero) eigenvalue is even.*

Since there is a bijection between orthogonal solutions of (1) and symmetric solutions of (5), we may use Theorem 8.4.3 in [11] to decide whether (1) has a finite or infinite number of orthogonal solutions. For the sake of completeness, we summarize that result below, after introducing some notation. For every $\lambda \in \sigma(\overline{\mathcal{H}})$, let $R_\lambda(\overline{\mathcal{H}})$ denote the corresponding generalized eigenspace and $m_g(\lambda)$ the geometric multiplicity of λ .

Theorem 3.10. *The number of orthogonal solutions of (1) is finite if and only if $m_g(\lambda) = 1 \forall \lambda \in \sigma(\overline{\mathcal{H}})$. In this case, the number of orthogonal solutions is given by*

$$\left(\prod_{j=1}^p (\dim R_{\lambda_j}(\overline{\mathcal{H}}) + 1) \right) \cdot \left(\prod_{j=1}^q \left(\frac{1}{2} \dim R_{\alpha_j \pm i\beta_j}(\overline{\mathcal{H}}) + 1 \right) \right),$$

where $\lambda_1, \dots, \lambda_p$ are all the distinct real positive eigenvalues of $\overline{\mathcal{H}}$, and $\alpha_1 \pm i\beta_1, \dots, \alpha_q \pm i\beta_q$ are all the distinct pairs of complex conjugate eigenvalues of $\overline{\mathcal{H}}$ having positive real parts.

Remark 3.11. For the particular case when $\overline{\mathcal{H}}$ has distinct eigenvalues, (1) has a finite (even) number of orthogonal solutions, half of which are special orthogonal.

Definition 3.12. We say that a subset $\Sigma \subset \mathbb{C} \setminus i\mathbb{R}$ admits a good splitting if $\Sigma = \Sigma_+ \cup \Sigma_-$, where $\Sigma_+ \cap \Sigma_- = \emptyset$ and both Σ_+ and Σ_- satisfy the properties (P1) and (P2) in (4).

Now assume that $\sigma(\overline{\mathcal{H}}) \cap i\mathbb{R} = \emptyset$. Under this assumption, the spectrum of $\overline{\mathcal{H}}$ always admits a good splitting. An example of such a splitting consists in considering Σ_- the set of all eigenvalues of $\overline{\mathcal{H}}$ in the open left half plane and Σ_+ the set of all eigenvalues in the open right half plane.

Theorem 3.13. *If $\overline{\mathcal{H}}$ has no pure imaginary eigenvalues, then for any good splitting of the spectrum of $\overline{\mathcal{H}}$, $\sigma(\overline{\mathcal{H}}) = \Sigma_+ \cup \Sigma_-$, there exists a unique orthogonal solution X of (1) with $\sigma(XJ) = \Sigma_+$.*

Proof. Under this spectral assumption, the equivalence between (i) and (iv) in Theorem 2.1 guarantees the existence of symmetric solutions of (5). And Theorem 2.3 ensures the existence of a unique symmetric solution S of (5) such that $\sigma(M/2 + S) = \Sigma_+$. Since $M/2 + S = XJ$, the result follows. \square

4. Comparison with the Moser–Veselov approach

In order to contrast our method with the Moser–Veselov approach in [13], we outline here their main steps. The first observation is that if X is orthogonal, Eq. (1) is equivalent to the quadratic matrix equation

$$W^2 - MW - J^2 = 0, \quad (8)$$

with the additional condition $W^T W = J^2$. Clearly, the relationship between solutions of Eqs. (1) and (8) is $X = WJ^{-1}$. Now, if λ is an eigenvalue of W , it follows from (8) that

$$\det(\lambda^2 I - \lambda M - J^2) = 0. \quad (9)$$

The main result in [13], concerning solutions of Eq. (1), is Theorem 1' in p. 228, which we may state in the following way.

Theorem 1'. *If the set Σ of the roots of Eq. (9) does not intersect the imaginary axis $i\mathbb{R}$, then for any good splitting $\Sigma = \Sigma_+ \cup \Sigma_-$, there exists a unique solution W of (8) (and therefore a unique solution of (1)) with $\sigma(W) = \Sigma_+$.*

Two observations have to be made in order to see the connection between this theorem and Theorem 3.13. First note that the characteristic polynomial of the matrix

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ J^2 & M \end{bmatrix},$$

used in [13] to prove Theorem 1', is equal to $\det(\lambda^2 I - \lambda M - J^2)$. Secondly, the matrix \mathcal{A} is similar to the Hamiltonian matrix $\overline{\mathcal{H}}$. Indeed, the matrix

$$P = \begin{bmatrix} I & 0 \\ M/2 & I \end{bmatrix}$$

is such that $P\overline{\mathcal{H}}P^{-1} = \mathcal{A}$. So, Theorem 1' in [13] is just a restatement of Theorem 3.13.

5. Computation of special orthogonal solutions

Based on our analysis, the computation of orthogonal or special orthogonal solutions of (1) depends on the computation of symmetric solutions of the associated algebraic Riccati equation. There are several numerical methods for this purpose, one of the most reliable being the Schur method [7,12], which is stable. Other reliable methods include the Newton method [8], the method of the matrix sign function [1] and symplectic QR-like methods [1,3,14], which take into account the particular structure of Hamiltonian matrices and are consequently less costly.

In general, these methods compute the unique non-negative definite solution (provided it exists) for the algebraic Riccati equation and require that \mathcal{M} has no imaginary eigenvalues. Under some assumptions, there are several sufficient and necessary/sufficient conditions that guarantee the existence of such a solution [9]. In our case, it is enough to assume that $M^2/4 + J^2 > 0$ to conclude that (5) has a unique non-negative definite solution that corresponds to a special orthogonal solution of (1).

Theorem 5.1. *Suppose that $M^2/4 + J^2$ is positive definite. Then:*

- (i) *Eq. (5) has a unique solution $S \geq 0$ such that the eigenvalues of $S + M/2$ have positive real parts.*
- (ii) *$X = (S + M/2)J^{-1}$, where S is the matrix of the previous (i), is special orthogonal.*

Proof

- (i) Since $M^2/4 + J^2$ is positive definite, there exists an invertible matrix E such that $M^2/4 + J^2 = E^T E$. We will show that $(M/2, I)$ is stabilizable and $(E, M/2)$ is detectable, so that part (i) follows from Theorem 2.4. Indeed, since the eigenvalues of $\pm M/2 - I$ are of the form $-1 \pm \alpha i$ ($\alpha \in \mathbb{R}$), they always have negative real part. Therefore both $(M/2, I)$ and $(-M/2, E^T)$ are stabilizable, that is, $(M/2, I)$ is stabilizable and $(E, M/2)$ is detectable.
- (ii) Immediate consequence of (i). \square

By the previous theorem together with (7) it is enough to assume that $M^2/4 + J^2 > 0$ to ensure that the unique non-negative definite solution of (5) can be computed and that \mathcal{H} has no pure imaginary eigenvalues.

We now suppose that $M^2/4 + J^2$ is non-negative definite instead of positive definite as above. In this case, \mathcal{H} may have pure imaginary eigenvalues, which occur, for example, when $J = I$ and

$$M = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

in (1). However, the existence of a special orthogonal solution is guaranteed by the next theorem.

Theorem 5.2. *If $M^2/4 + J^2 \geq 0$, then Eq. (1) admits a special orthogonal solution.*

Proof. By Theorem 5.1, there exists a solution $S \geq 0$ for Eq. (2). It remains to show that the real eigenvalues of $M/2 + S$ are positive. In fact, if λ is a real eigenvalue of such a matrix, then it admits an associated real eigenvector u . Since

$$0 \leq u^T S u = u^T (M/2 + S) u = u^T \lambda u = \lambda u u^T$$

it follows that $\lambda = (u^T S u)/(u^T u) \geq 0$. Since $M/2 + S$ is invertible, we have $\lambda \neq 0$, and therefore $\lambda > 0$. \square

For the sake of completeness we outline the main steps of an algorithm to compute the special orthogonal matrix $X = (S + M/2)J^{-1}$ using the Schur method of [12].

Algorithm. Assume that $M^2/4 + J^2$ is positive definite. This algorithm computes a special orthogonal solution X of (1).

1. Find a real Schur form of \mathcal{H} ,

$$R^T \overline{\mathcal{H}} R = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

such that the real parts of the spectrum of H_{11} are negative and the real parts of the spectrum of H_{22} are positive, and partition R conformably into four blocks

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

2. Compute $S = R_{21} R_{11}^{-1}$. (R_{11} is invertible and $S^T = S \geq 0$ is a solution of (5)).
3. Compute $X = (S + M/2)J^{-1}$.

6. A particular case of the Moser–Veselov equation

While Eq. (1) is associated to the discrete model of the dynamics of a generalized rigid body, the continuous model is associated to the following algebraic matrix equation:

$$YQ^T - QY^T = M, \quad (10)$$

where Q orthogonal and M skewsymmetric are given. The objective now is to find orthogonal solutions of (10). Introducing $X = YQ^T$, this equation reduces to

$$X - X^T = M, \quad (11)$$

which is a particular case of Eq. (1), with $J = I$. So, the previous analysis may be used. However, in this case important simplification occur which leads to explicit formulas for the corresponding solutions. Also in this case, there exists a necessary and sufficient condition for the existence of orthogonal solutions of (11), in terms of the spectrum of M .

Theorem 6.1. Every orthogonal solution of (11) can be written in the form

$$X = \frac{M}{2} + \left(\frac{M^2}{4} + I \right)^{1/2},$$

where $\left(\frac{M^2}{4} + I \right)^{1/2}$ denotes any symmetric square root of $\frac{M^2}{4} + I$ that commutes with M .

Proof. We first note that, if X is an orthogonal solution of (11), then X commutes with M . Indeed, $MX = (X - X^T)X = X^2 - I$ and $XM = X(X - X^T) = X^2 - I$.

Using Theorem 3.3 we know that if X is an orthogonal solution of (11), then $X = S + M/2$, where S is a symmetric solution of the Riccati equation

$$S^2 + S(M/2) + (M/2)^T S - (M^2/4 + I) = 0.$$

But since X commutes with M , also S commutes with M and the previous equation reduces to

$$S^2 - (M^2/4 + I) = 0.$$

This implies the result. \square

Theorem 6.2. *The matrix equation (11) has special orthogonal solutions if and only if $\sigma(M) \subset [-2i, 2i]$.*

Proof: (\Rightarrow) This follows from the canonical real Jordan forms of skew-symmetric and orthogonal matrices (see [6] for details). Indeed, if M is skew-symmetric, there exists an orthogonal V such that

$$M = V \operatorname{diag}(M_1, \dots, M_k, \mu_1, \dots, \mu_s) V^T,$$

where

$$M_j = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}, \quad \alpha_j \in \mathbb{R}^+, \quad j = 1, \dots, k,$$

and $\mu_i = 0$, $i = 1, \dots, s$. Also, if X is an orthogonal solution of (11), then there exists U orthogonal such that

$$X = U \operatorname{diag}(X_1, \dots, X_k, \lambda_1, \dots, \lambda_s) U^T,$$

where

$$X_j = \begin{bmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{bmatrix}, \quad \theta_j \in \mathbb{R}, \quad j = 1, \dots, k,$$

and $\lambda_i = \pm 1$, $i = 1, \dots, s$. Replacing this expressions of M and X in (11), a simple calculation shows that $\sigma(M) \subset [-2i, 2i]$.

(\Leftarrow): We will show that, if $\sigma(M) \subset [-2i, 2i]$, then there exists a square root of $M^2/4 + I$ such that

$$X = \frac{M}{2} + \left(\frac{M^2}{4} + I \right)^{1/2}$$

is special orthogonal. Indeed, under this spectral condition on M , the principal matrix square root, i.e. the unique square root with eigenvalues in the open right half plane, here denoted by $\sqrt{M^2/4 + I}$, is symmetric and commutes with M . Besides, the orthogonal matrix

$$Y = \frac{M}{2} + \sqrt{\frac{M^2}{4} + I}$$

is special because its eigenvalues are of the form $\lambda = \frac{\alpha}{2}i \pm \sqrt{1 - \alpha^2/4}$, where $\alpha i \in \sigma(M)$. \square

As far as we know, the matrix equation (10) appeared first in [2], associated to the dynamics of the generalized rigid body. Special orthogonal solutions of (10) were given in terms of the inverse matrix hyperbolic sine. Numerical considerations about computing such solutions were presented in [4]. We now contrast the results presented above with those obtained previously.

Assume that X is a special orthogonal solution of (11). Then, there exists a skew-symmetric matrix A such that $X = e^A$. Replacing in (11) we obtain

$$e^A - (e^A)^T = M \Leftrightarrow \frac{e^A - e^{-A}}{2} = \frac{M}{2} \Leftrightarrow A = \sinh^{-1}(M/2).$$

We now have to use a result proved in [4] which states the following:

If C is skew-symmetric, then the matrix equation $\sinh X = C$ has a skew-symmetric solution if and only if $\sigma(C) \subset \{\alpha i : -1 \leq \alpha \leq 1\}$.

Clearly this agrees with the spectral condition in Theorem 6.2. We have also proved that, in this case, $\text{Sinh}^{-1}(M/2)$, where Sinh^{-1} denotes the principal inverse hyperbolic sine, is a well defined primary matrix function given by

$$\text{Sinh}^{-1}(M/2) = \text{Log} \left(\frac{M}{2} + \sqrt{\frac{M^2}{4} + I} \right),$$

where $\text{Log}(\cdot)$ stands for the principal matrix logarithm (see [5] for more details about primary matrix functions).

So, a special orthogonal solution of (11) may be written as

$$X = e^{\text{Sinh}^{-1}(M/2)} = e^{\text{Log} \left(\frac{M}{2} + \sqrt{\frac{M^2}{4} + I} \right)} = \frac{M}{2} + \sqrt{\frac{M^2}{4} + I},$$

which agrees with the construction in the proof of Theorem 6.2. However, the approach in [4], which is based on the assumption that $Y = e^A$, for some skew-symmetric matrix A , is less general than the method presented here.

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